Grading guide, Pricing Financial Assets, June 2020

1. Consider a stock that does not pay dividends during the period under analysis. Let the price be S_t at time t.

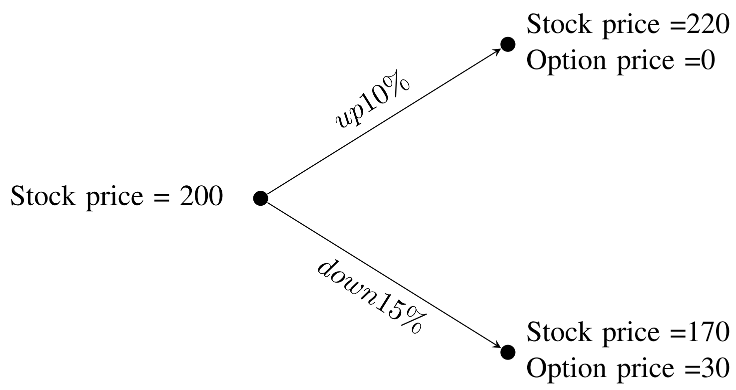
   At time 0 the stock price is 200, and during the next year the price can go up by 10% or down by 15%. The one year risk free interest rate is 0.

   a) Find the arbitrage free price p_0 at time 0 of a put option with strike 200 and expiry after one year by constructing a risk free portfolio of the stock and the option.

   b) What is the risk neutral probability q of the up-move in the stock price?

   c) Suppose that the real world probability p of an up-move is $4/5$. Why would you expect $p > q$? What is the expected return on the put option? Explain the result.

Solution: The answer can be found using the binomial model, ref the illustration below:



a) Consider a portfolio of 1 putoption and long Δ stocks. The payoffs at time 1 will be 220Δ and $170\Delta + 30$. This is risk free if Δ is chosen to $3/5$ and will give a payoff of 132 at time 1. That amount can be borrowed at time 0 giving a cash-flow of 132. Deduct the price of the delta position in the stock of 120 and you have 12 left. This must be the price of the put if there is to be no arbitrage opportunities.

b) The risk neutral probability q of the up-move can be found by $200 = 220q + 170(1 - q)$. Thus $q = 3/5$. That provides another way to calculate the arbitrage free put price, since the discounted expected payoff is $30 * (1 - 3/5) = 12$

c) The scenario may be rationalised by the following: The real world probability of a state that is "good", most likely with high stock prices, is higher than the risk neutral probability, since the risk neutral probability will correct for a risk premium penalizing payoffs in "good" states and benefiting payoffs in "bad" states. The expected return of the put option under the real world probability is negative ((30/5 -12)/12=-50%) and lower than the risk free rate (of 0%) since the put can serve as an insurance against the "bad" down-state. Note that the real world expected return of the stock is (220*4/5+170/5)/200-1 = 5%, much higher than the risk free rate, thus indicating a high risk premium.

2. In a tranched CDS (also a synthetic CDO) compare the most senior, the mezzanine and the most junior (also called "equity") tranches.

   a) Explain how the tranches are exposed to the risk of defaults from the total underlying portfolio. Explain what is meant by "attachment point" and "detachment point".
   b) Assume a large set of obligors (or "names") in the underlying portfolio. If correlation is not perfect which tranche will gain proportionally the most by a lower credit spread on the underlying portfolio?
   c) For a given credit spread on the underlying portfolio of CDS which tranche will benefit from an increase in implied (i.e. derived from market prices) correlation?

Solution:

a) The equity tranche can be seen as a leveraged exposure to the total underlying portfolio, but protected by limited liability. The more senior tranches are exposed to the risk on the total underlying portfolio only after more junior tranches has covered losses. The attachment and detachment points of a given tranche denotes the proportions of losses on the underlying portfolio that is covered by the given tranche, e.g. the equity tranche has an attachment point of 0 and a detachment point of e.g. 3%, whereas the most senior tranche has an attachment point of e.g. 22% and a detachment point of 100%.

b) In general the spread on the junior tranche is higher than on the senior tranche as it has a higher expected loss from defaults in the underlying and a higher risk (the effect of leverage). The price of the most junior tranche will benefit the most (unless continuing to be worthless) from a fall in credit spreads.

c) An increase in implied correlation (for a given spread) will benefit the equity tranche and hurt the most senior tranche. To see this compare the extremes where defaults are perfectly correlated to where they are independent: First note that the underlying portfolio and thus the sum of tranche values stay fixed (as the spread is the same). If defaults are perfectly correlated the (risk neutral) expected loss reflected in the given spread will take the form of no losses *or* of losses that will hit the equity tranche but likely also spill into the more senior tranches. If defaults are independent through the diversification (of the large set of underlying obligors) the losses will be close to the expected loss reflected in the given spread, hitting the equity tranche and maybe also spill into more senior tranches. So the price of the equity tranche will increase when the implied correlation increases. The price of the most senior tranche will fall (or stay the same if the subordination is so large that it will not suffer losses in the perfect correlation case). The change of the price of mezzanine trances cannot be established without further assumptions.

3. Consider an economy with a (continuously compounded) risk free interest rate of r and a stock with price S_t at time t. Suppose that the stock pays a deterministic continuous dividend rate of δ.

   a) What is the forward price at time t on the stock for maturity $T > t$?
   b) Suppose that a forward contract on the stock with forward price K exists. What will the value $V(S_t, t; K)$ of that contract be at time t?
   c) The Black-Scholes-Merton partial differential equation for the price V of some general derivative is

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \tfrac{\partial^2 V}{\partial S^2} + (r-\delta)S\frac{\partial V}{\partial S} - rV = 0$$

Show that the value of the forward contract satisfies this equation.

Solution:

a) The value can be found taking the risk neutral expectation of the payoff at T from a strategy where dividends are continuously reinvested in the stock and discounting with the risk free rate: To achieve a value of S_T at T invest $S_t \mathrm{e}^{-\delta(T-t)}$ in the stock at time t and reinvest dividends (the value will grow proportional to $dS_t + \delta dt$). The forward price will be a value F_t that satisfy:

$$
\begin{aligned}
0 &= \mathrm{e}^{-r(T-t)} E_t^{\mathbb{Q}}[S_T - F_t] \\
\mathrm{e}^{-r(T-t)} F_t &= \mathrm{e}^{-r(T-t)} E_t^{\mathbb{Q}}[S_T] \\
\mathrm{e}^{-r(T-t)} F_t &= S_t \mathrm{e}^{-\delta(T-t)} \\
F_t &= S_t \mathrm{e}^{(r-\delta)(T-t)}
\end{aligned}
$$

b) The value of a forward contract with a given forward price K will be

$$
V(S_t, t; K) = \mathrm{e}^{-r(T-t)} E_t^{\mathbb{Q}}[S_T - K] = S_t \mathrm{e}^{-\delta(T-t)} - K \mathrm{e}^{-r(T-t)}
$$

c) Taking partials it can be shown that this satisfies the BSM PDE.